

Existence of Orbits in Simple Mechanical Systems

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ABSTRACT. A simple mechanical system consists of a Riemannian manifold, $(M, \langle \cdot, \cdot \rangle)$ and a smooth potential function, $V : M \rightarrow \mathbb{R}$. There is a natural second order Hamiltonian system that is associated with such a triple. Using techniques from the calculus of variations on infinite dimensional manifolds, we will prove the existence of subharmonic, homoclinic, and heteroclinic orbits for a large class of simple mechanical systems.

Introduction. A simple mechanical system consists of a compact Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ and a smooth potential function $V : M \rightarrow \mathbb{R}$. As an example, consider the sphere with its metric of constant curvature together with a height function. A given system gives rise to the following equations of motion:

$$(HS) \quad \nabla_{\gamma'} \gamma' = -\text{grad}_M V(\gamma)$$

where ∇ denotes the Levi-Cevita connection for $(M, \langle \cdot, \cdot \rangle)$. If the potential is constant, (HS) reduces to the equations satisfied by geodesics on the given manifold.

The geodesic equations are one of the most basic differential equations in geometry. One of the nice properties that it enjoys is its invariance under a rescaling of the time parameter: if $\gamma(t)$ is a solution to the geodesic equations, then the curve $\tilde{\gamma}(t) = \gamma(ct)$ satisfies the same equations. So in the case of geodesics one can assume that the curve is parametrized on the unit interval. To prove existence of geodesics, one can proceed by approximating the space of curves with a finite-dimensional manifold (see [Mi]). An alternative approach is to think of the space of curves as a manifold modelled on a Hilbert space and the geodesics as critical points of a function on this manifold. We will adopt the latter point of view in our study of orbits in mechanical systems.

The equations (HS) behave differently when we rescale the time parameter: if $\gamma(t)$ is a solution to (HS), then the curve $\tilde{\gamma}(t) = \gamma(ct)$ satisfies the equation

$$\nabla_{\tilde{\gamma}'} \tilde{\gamma}' = -c^2 \text{grad}_M V(\tilde{\gamma}).$$