

# Smoothness at Infinity

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**ABSTRACT.** It is proved that a connected analytic subvariety of  $\mathbb{C}^n$  that is smooth at infinity is a translate of a linear subspace of  $\mathbb{C}^n$ , and that an automorphism of  $\mathbb{C}^n$  that fixes the origin is smooth at infinity if and only if it is a multiple of a unitary transformation.

While trying to determine whether every periodic automorphism of  $\mathbb{C}^n$  is conjugate to a linear one we became aware of some topological results concerning periodic diffeomorphisms of spheres (such as Proposition 6 in [1]) which looked as if they could shed some light on our problem. They could of course be relevant to only those automorphisms of  $\mathbb{C}^n$  that are smooth at infinity, and this led us to ask: Which automorphisms  $F$  of  $\mathbb{C}^n$  are smooth at infinity?

This question has a (to us) surprisingly simple answer, given by Theorem I of the present paper.

Our first proof of that result was based on an investigation of the zero-sets of the components of  $F$ . We thus derived Theorem I from a special case of what is now Theorem II. The latter characterizes those analytic subvarieties of  $\mathbb{C}^n$  that are smooth at infinity. It was only later that we found the extremely simple proof of Theorem I which is given here.

We define smoothness at infinity by means of the inversion

$$(1) \quad x \rightarrow \tau(x) = \frac{x}{|x|^2}$$

where  $|x|$  is the euclidean distance from the origin to  $x \in \mathbb{C}^n$  (or  $x \in \mathbb{R}^N$ ). It is understood that  $\tau(0) = \infty$ ,  $\tau(\infty) = 0$ , so that  $\tau$  is a homeomorphism of the one-point compactification of  $\mathbb{C}^n$ , or  $\mathbb{R}^N$ .

Two obvious properties of  $\tau$  are

$$(2) \quad \tau^{-1} = \tau, \quad |\tau(x)| = 1/|x|.$$

**Definition.** A map  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is said to be  $\mathcal{C}^1$ -smooth at infinity if its conjugate

$$(3) \quad \Phi = \tau^{-1}F\tau,$$

defined on  $\mathbb{C}^n \setminus \{0\}$ , extends to  $\mathbb{C}^n$  so as to be continuously differentiable in a neighborhood of the origin.