

Recapturing Functions in H^p Spaces

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1. Introduction. Let D be a domain in \mathbf{C}^n and let X be a linear metric space of functions analytic in D . We call a sequence $\{\ell_n\}_{n=1}^\infty$ of elements of the dual space X^* a uniqueness sequence if $\ell_n(f) = 0$ for $n = 1, 2, \dots$ implies $f = 0$. The more general notion of uniqueness sets for X is defined in a similar way. The simplest type of uniqueness sequence occurs when a sequence $A = \{a_n\}_{n=1}^\infty \subset D$ has the property that zero is the only function in X that vanishes on A . Thus if $D = \Delta \equiv \{z : |z| < 1\}$ and $X = H^p = H^p(\Delta)$, then any sequence which does not satisfy the Blaschke condition, that is one for which $\sum_{n=1}^\infty (1 - |a_n|) = \infty$, yields a uniqueness sequence. The functionals ℓ_n in this example are point evaluations at a_n , that is, $\ell_n(f) = f(a_n)$ for $f \in H^p$.

A more general way of obtaining uniqueness sequences is from complex measures on D or on \bar{D} , say, when the functions in X have boundary values. Thus we let $\ell_n(f) = \int_{\bar{D}} f(z) d\mu_n(z)$. Suppose that D is a bounded strictly pseudoconvex domain in \mathbf{C}^n and X is the Hardy space $H^p(D)$. Then subsets E of ∂D for which the $(2n-1)$ -Hausdorff measure $h_{2n-1}(E) > 0$ give uniqueness sets for point evaluations on E . This gives a uniqueness sequence in the following way. Let $U_n = \{V_j^n\}_{j=1}^{m_n}$ be an open covering of ∂D by balls of radius $1/n$. Set $E_{nk} = E \cap V_k^n$ and let $\ell_{nk}(f) = \int_{E_{nk}} f(z) dh_{2n-1}(z)$ for $k = 1, 2, \dots, m_n$ and $n = 1, 2, \dots$. Then $\{\ell_{nk}\}$ is the desired sequence.

We consider the question of recapturing an analytic function f from its values on a uniqueness sequence. More precisely, given $b_n = \ell_n(f)$ ($n = 1, 2, \dots$) for some $f \in X$, we seek to construct a sequence $\{f_n\}$ of elements of X which converges to f in X . The investigation of such problems was started long ago, mostly for functions analytic in Δ . One of the first results was obtained by T. Carleman [2]. Using the harmonic measure of a set $E \subset \partial\Delta$ having positive Lebesgue measure (with respect to the unit circle) and values of $f \in H^p$ on E , he constructed a sequence of functions in H^p which converged to f uniformly on compacta. There are many results about uniform convergence on compacta of polynomial and rational interpolants. A detailed bibliography on the subject can be found in [10]. More recently, D. J. Patil [9] modified Carleman's method and, by only using values of f on E , his construction yields a sequence which