

Sobolev-Poincaré Inequalities for $p < 1$

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ABSTRACT. If Ω is a John domain (or certain more general domains), and $|\nabla u|$ satisfies a certain mild condition, we show that $u \in W_{\text{loc}}^{1,1}(\Omega)$ satisfies a Sobolev-Poincaré inequality

$$\left(\int_{\Omega} |u - a|^q \right)^{1/q} \leq C \left(\int_{\Omega} |\nabla u|^p \right)^{1/p}$$

for all $0 < p < 1$, and appropriate $q > 0$. Our conclusion is new even when Ω is a ball.

1. Introduction. The (q,p) -Poincaré (or Sobolev-Poincaré) inequality

$$(1.1) \quad \inf_{a \in \mathbf{R}} \left(\int_{\Omega} |u - a|^q \right)^{1/q} \leq C \left(\int_{\Omega} |\nabla u|^p \right)^{1/p}$$

is known to be true for all $u \in W^{1,p}(\Omega)$ and $1 \leq p < n$, $0 < q \leq np/(n-p)$, if Ω is a bounded Lipschitz domain, or even a John domain (see [Boj], [M]). There are simple examples which show that this inequality is false for all $p < 1$, even if q is very small, Ω is a ball, and u is smooth (one such example is given near the end of Section 1). Nevertheless, we shall show that, under a rather mild condition on ∇u , one can prove such an inequality in any John domain for all $0 < p < 1$ (see Theorem 1.5).

Presumably because of the simple counterexamples, there has been almost no previous research on Poincaré-type inequalities for $p < 1$. One notable exception is to be found in [K], where the assumptions (quasiconformality and $0 < q < np/(n-p)$) are stronger than in Theorem 1.5.

We shall state the first version of the Poincaré inequality in Theorem 1.5, and prove it in Section 3 after a couple of preparatory lemmas in Section 2. Section 4 contains a weak “converse” to Theorem 1.5 and, finally, Section 5 contains some related results.

Let us first introduce some necessary notation and terminology. A cube Q is always assumed to have faces perpendicular to the coordinate directions, $\ell(Q)$