

Superattractive Fixed Points in \mathbf{C}^n

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1. Introduction. Many of the most important algorithms of mathematics are iterative, and in the good cases quadratically convergent. The most typical of these is Newton's method. If x_0 is a root of the equation $f(x) = 0$ and $f'(x_0) \neq 0$, then if you start iterating Newton's algorithm at some y_0 sufficiently close to x_0 , generating the sequence y_0, y_1, y_2, \dots , you can expect the errors $E_n = |y_n - x_0|$ to satisfy approximately $E_{n+1} \approx (E_n)^2$, so that $E_n \approx (E_0)^{2^n}$.

Note that this is very much more rapid than geometric convergence, where $E_n \approx \alpha^n E_0$ for some α satisfying $0 < \alpha < 1$: in one case you double the number of correct digits, in the other you add approximately $-\log_m \alpha$ correct digits in base m at each iteration.

A fixed point x_0 of a mapping f will be called superattractive if there is a neighborhood U of x_0 such that

$$|f^{\circ n}(y) - x_0| \leq C|f^{\circ(n-1)}(y) - x_0|^2$$

for some constant C , and all $n > 0$, $y \in U$.

In this paper we will try to analyze the behavior of an analytic mapping f near a superattractive fixed point x_0 ; the condition above shows that the linear terms of f vanish at x_0 .

In dimension one, there is a very clean way of saying essentially all there is to say about the local behavior of f near x_0 . If $f(x_0 + u) = x_0 + au^k + \dots$ with $a \neq 0$, then there is an analytic local coordinate φ at x_0 such that $\varphi(f(x)) = \varphi(x)^k$. The local coordinate is called the *Böttcher coordinate*; it is unique up to multiplication by a $(k-1)^{\text{th}}$ root of 1 [M and DH].

The analogous statement is false in higher dimensions. Let $f : U \rightarrow \mathbf{C}^n$ be an analytic map defined on an open subset $U \subset \mathbf{C}^n$, and x_0 a superattractive fixed point of f . When $n \geq 2$, the map f is not in general locally conjugate, even topologically, to its terms of lowest degree; the local geometry near such a point is much too rich for anything like that to be true.

To see this, consider the case $n = 2$, and suppose $x_0 = 0$. Generically such a mapping f has near 0 a critical locus which consists of two transversal non-singular curves. Since the critical locus is exactly the set of points at which the