

Liouville Theorems for Even Order Elliptic Systems

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1. Introduction. We consider solutions defined in all of \mathbb{R}^n of the complex vector differential equation of even order $L = 2\ell$, $\ell \geq 1$,

$$(1.1) \quad \mathcal{L}U := \sum_{|\alpha| \leq L} A_\alpha D^\alpha U = 0.$$

Each coefficient A_α is an $m \times m$ complex matrix valued function on \mathbb{R}^n , and the solution U an $m \times 1$ complex vector in the space $\mathcal{H}_{\text{loc}}^L(\mathbb{R}^n)$. We present certain decay and smoothness conditions on the coefficients, and “strongly coercive” and ellipticity conditions on the operator, which ensure the validity of the statements:

- (a) Solutions of polynomial growth are necessarily identically zero.
- (b) Solutions of polynomial growth are necessarily polynomials.

In [27], N. Weck, for the case of constant coefficients, established the following assertions:

- (c) Bounded solutions of (1.1) must be constant iff the matrix

$$S(x) := \sum_{|\alpha| \leq L} A_\alpha x^\alpha$$

is invertible for all x in $\mathbb{R}^n \setminus \{0\}$.

- (d) Bounded solutions of (1.1) must be identically zero iff $S(x)$ is invertible for all x in \mathbb{R}^n .

Although Weck stated these results for equations of even order, an examination of his proof reveals that it is equally valid for odd order equations. (It is known that elliptic operators with real coefficients must be of even order. However, for operators with complex coefficients, the order can be odd.)

For the case of variable coefficients and equations of even order, Weck gave sufficient conditions under which bounded solutions are necessarily identically