

Viscosity Solutions of Second Order Fully Nonlinear Elliptic Equations with State Constraints

MARKOS A. KATSOULAKIS

ABSTRACT. This work is concerned with *State Constraints* boundary value problems for second order fully nonlinear degenerate equations. Existence, uniqueness and regularity questions are addressed for viscosity solutions which are required to be supersolutions on the boundary of a domain, this last property playing the role of a boundary condition. Such problems arise when a functional is minimized over controlled, stochastic dynamics, constrained inside a domain *without* using reflection on its boundary.

Introduction. In this paper we study the class of viscosity solutions of the second order *Bellman* equation in a domain $\Omega \subset \mathbb{R}^n$ which are required to be viscosity supersolutions on $\partial\Omega$. More precisely, we seek functions $u : \bar{\Omega} \rightarrow \mathbb{R}$ such that

$$(SC) \quad \begin{cases} u + \sup_{a \in U} \{-\text{tr } A(x, a) D^2 u - b(x, a) Du - h(x, a)\} = 0 & \text{in } \Omega \\ u + \sup_{a \in U} \{-\text{tr } A(x, a) D^2 u - b(x, a) Du - h(x, a)\} \geq 0 & \text{on } \partial\Omega \end{cases}$$

(U is a compact metric space and Du, D^2u are the first and second order differentials of u).

This last property of u on $\partial\Omega$ is not pointwise, as we explain in Section 1, and plays essentially the role of a boundary condition for the Bellman equation. Such problems have been studied by H. M. Soner, P. Loretto, I. Capuzzo-Dolcetta and P. L. Lions in the context of first order Hamilton-Jacobi equations. For more details on their work see [S], [Lo], [CdL].