

# *Volumes of Hyperbolic 3-Orbifolds*

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T. Jørgensen and W. Thurston [Th1, Chapters 5 and 6] developed a structure theory for the set of volumes of complete, oriented, finite-volume hyperbolic 3-manifolds. They proved that volumes decrease when (hyperbolic) Dehn surgery is performed on a cusp of the manifold. Also, the usual order on the real numbers induces a well-ordering on the set of volumes, and there are at most finitely many (isometry classes of) hyperbolic 3-manifolds having any fixed volume.

In this paper we will generalize these theorems to encompass quotient spaces of hyperbolic 3-manifolds by finite groups of isometries, or more precisely, complete, oriented, finite-volume hyperbolic 3-orbifolds. We are motivated both by the central role which orbifolds play in the geometric attack on the classification of 3-manifolds (as seen in the proof of the existence of hyperbolic structures on Haken atoroidal 3-manifolds, due to W. Thurston [Th2]) and by recent results concerning low-volume hyperbolic 3-orbifolds (and 3-manifolds). For instance, in [M1], the cusped hyperbolic 3-orbifold of minimum volume is found. T. Chinburg and E. Friedman [CF] have found the arithmetic hyperbolic 3-orbifold of minimum volume. C. Adams [A1, A2] has found the six lowest volume cusped hyperbolic 3-orbifolds. In [M2], a lower bound for the volume of a hyperbolic 3-orbifold was obtained, and this bound was recently improved upon by F. Gehring and G. Martin [GM]. The volumes of arithmetic hyperbolic 3-orbifolds have also been studied by A. Borel [Bo] and by C. Maclachlan and A. Reid [MR]. The proper context within which to place the papers above is a structure theory for the set of volumes of hyperbolic 3-orbifolds. This structure is provided here.

**1. Basic definitions.** A (smooth) *orbifold*  $Q$  is a topological space which locally has the structure of a quotient space of  $\mathbb{R}^n$  by a (smooth) finite group action. Its *singular set*  $\Sigma_Q$  corresponds to the fixed points of these local actions. Examples when  $n = 2$  include *footballs*, *turnovers*, and *pillows*, which are 2-orbifolds which are topologically 2-spheres, and which have singular set consisting of (respectively) 2, 3, and 4 points, each of which is modelled on  $\mathbb{R}^2$  modulo a cyclic group acting by rotations. It is often useful to specify the orders of these cyclic groups, and refer to a certain 2-orbifold as a (2,3,4)-turnover, for example. When  $n = 3$  and the orbifold is orientable, the singular set is always a trivalent graph, and the orbifold is topologically a 3-manifold. If  $\Gamma$  is a discrete subgroup