

Locally Conformally Flat Four- and Six-Manifolds of Positive Scalar Curvature and Positive Euler Characteristic

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ABSTRACT. In this paper, we prove that a compact 4- or 6-dimensional manifold which is locally conformally flat with positive scalar curvature and positive Euler characteristic is conformally equivalent to the sphere or projective space. Our proof includes solving the Yamabe problem in this setting but does not require the positive mass theorem. We also prove an L^2 pinching theorem for the Weyl curvature of a 4-manifold with positive scalar curvature and positive Euler characteristic.

Introduction and Statement of Results. A compact surface (M, g) with positive scalar curvature must have Euler characteristic $\chi(M) > 0$ by the Gauss-Bonnet formula. Then, by the classification of surfaces, the diffeomorphism type of M is determined (M is either S^2 or projective space $\mathbb{R}P^2$), and finally, by the uniformization theorem, one concludes that g is conformally equivalent to the canonical metric of constant curvature.

This classical result indicates the power of the Gauss-Bonnet formula for surfaces, which serves as a “bridge” between the topological and geometric information. While the situation in higher dimensions is naturally more complicated, in this paper we show that an analogous result does hold in four and six dimensions (but not in dimensions greater than six). Aside from its intrinsic interest, this result has an interesting connection to the Yamabe problem which we will discuss.

Theorem A. *Let M be a compact 4-or 6-dimensional manifold which admits a locally conformally flat metric of non-negative scalar curvature \bar{g} . Then $\chi(M) \leq 2$. Furthermore, $\chi(M) = 2$ if and only if (M, \bar{g}) is conformally equivalent to the sphere with its canonical metric, and $\chi(M) = 1$ if and only if (M, \bar{g}) is conformally equivalent to projective space with its canonical metric.*