

Closed Geodesics in 2-Step Nilmanifolds

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ABSTRACT. We study the behaviour of smoothly closed unit speed geodesics in 2-step nilmanifolds. Let Γ be a lattice in a simply connected 2-step nilpotent Lie group N . The 2-step nilmanifold $\Gamma \backslash N$ is said to have density of closed geodesics if the vectors tangent to smoothly closed unit speed geodesics in $\Gamma \backslash N$ are dense in $S(\Gamma \backslash N)$, the unit tangent bundle. For non-singular N with center of arbitrary dimension, we show that an algebraic condition on the Lie algebra called resonance ensures density of closed geodesics for $\Gamma \backslash N$ where Γ is any lattice in N . We outline a method for constructing 6-dimensional examples and obtain a partial converse to the main result.

0. Introduction. The focus of this research is on the behaviour of geodesics in a simply connected, 2-step nilpotent Lie group endowed with a left invariant metric. By definition, 2-step nilpotent Lie groups are the nonabelian Lie groups that come as close as possible to being abelian. As such, they admit many interesting properties that do not appear in abelian Lie groups. If a simply connected, 2-step nilpotent Lie group is equipped with a left invariant metric then, following the philosophy of Milnor [Mi], the geometry of the Lie group can be studied by investigating the algebraic structure of its associated Lie algebra.

Let \mathcal{N} be a 2-step nilpotent Lie algebra whose unique, simply connected, 2-step nilpotent Lie group is N . Define an inner product \langle, \rangle on \mathcal{N} , which then extends to a left invariant metric on N . Let \mathcal{Z} be the center of \mathcal{N} and define $\mathcal{V} = \mathcal{Z}^\perp$, the orthogonal complement of \mathcal{Z} under the inner product \langle, \rangle .

For a 2-step nilpotent Lie algebra $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, Aroldo Kaplan [K1, K2] described a set of skew-symmetric maps $\{j(Z) : Z \in \mathcal{Z}\}$ such that $j(Z) : \mathcal{V} \rightarrow \mathcal{V}$ is defined by $j(Z)X = (\text{ad } X)^*Z$ for $Z \in \mathcal{Z}$, $X \in \mathcal{V}$, where ad is the adjoint representation on \mathcal{V} and $*$ is the metric adjoint. An equivalent (and often more useful) characterization is $\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$ for all $X, Y \in \mathcal{V}$, $Z \in \mathcal{Z}$.