

# Variation of Real Powers of Positive Operators

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ABSTRACT. For each real number  $p$  consider the map  $f(A) = A^p$  on the set of positive operators in a Hilbert space. Let  $Df(A)$  denote the Fréchet derivative of this map at  $A$ . In an earlier paper it was shown that  $\|Df(A)\| = \|pA^{p-1}\|$  when  $0 \leq p \leq 1$ . Here we show that this is true also when  $p < 0$  and when  $p \geq 2$  but not when  $1 < p < \sqrt{2}$ . Some other functions are also discussed.

**Introduction.** Perturbation bounds for functions of positive Hilbert space operators have been of interest to operator theorists, physicists and numerical analysts. They have analysed this problem in considerable detail paying special attention to the square root and other powers. A sample of this work can be obtained from the papers [1, 2, 5, 11, 12, 14].

Let  $f$  be a map of the positive half line  $(0, \infty)$  into itself and denote by  $f$  also the map this induces on the space  $\mathcal{B}_+$  of positive operators on a Hilbert space  $\mathcal{H}$ . Suppose this map is (Fréchet) differentiable [8] and let  $Df(A)$  denote the derivative of  $f$  at a point  $A$  of  $\mathcal{B}_+$ . This is a linear map on the space  $\mathcal{B}_s$  of self-adjoint operators; its action is given as:

$$(1.1) \quad Df(A)(B) = \left. \frac{d}{dt} \right|_{t=0} f(A + tB).$$

We have

$$(1.2) \quad \|Df(A)\| = \sup\{\|Df(A)(B)\| : B \in \mathcal{B}_s, \|B\| = 1\}.$$

Estimates of  $\|Df(A)\|$  lead to first order perturbation bounds for  $f$ .

Let  $f'$  denote the (ordinary) derivative of the function  $f$  on  $(0, \infty)$ . We then have

$$(1.3) \quad \|f'(A)\| = \|Df(A)(I)\|,$$