

Statistical Solutions of Stochastic Navier–Stokes Equations

MAREK CAPIŃSKI & NIGEL J. CUTLAND

ABSTRACT. Existence of statistical solutions to general stochastic Navier–Stokes equations with feedback in forces and the noise term is obtained without referring to the individual solutions to the equations. The approach employs nonstandard analysis to represent the measures involved by nonstandard densities.

Introduction. A general stochastic Navier–Stokes system with feedback in forces and noise has the form

$$(1) \quad \begin{cases} du = [\nu \Delta u - \langle u, \nabla \rangle u + f(t, u) + \nabla p] dt + g(t, u) dw_t, \\ \operatorname{div} u = 0, \end{cases}$$

where w is an infinite-dimensional Wiener process and the process $u(t, \omega)$ takes values in a suitable Hilbert space H . Heuristically, it is easy to derive the following equation for the family $(\mu_t)_{t \geq 0}$ of probability distributions of a solution $u(t)$ to (1): for a suitable test functional $\varphi(u)$,

$$(2) \quad \begin{aligned} & \int_H \varphi(u) d\mu_t(u) \\ &= \int_H \varphi(u) d\mu_0(u) + \int_0^t \int_H \left(-\nu \langle (u, \varphi'(u)) \rangle - b(u, u, \varphi'(u)) \right. \\ & \quad \left. + (f(s, u), \varphi'(u)) + \frac{1}{2} \operatorname{tr}(Qg^T(s, u)\varphi''(u)g(s, u)) \right) d\mu_s(u) ds \end{aligned}$$

where $b(u, u, v) = \langle (u, \nabla u), v \rangle_H$, Q is the covariance matrix of w , g^T denotes the transpose of the operator g , and φ' , φ'' are the Frechet derivatives of a sufficiently