

Some Results about the Existence of a Second Positive Solution in a Quasilinear Critical Problem

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ABSTRACT. Consider the problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{q-2} u + |u|^{p^*-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, $1 < q < p < N$, $\lambda > 0$, $p^* = Np/(N-p)$.

In this work we prove the existence of λ_0 such that for $0 < \lambda < \lambda_0$, the problem (P_λ) has at least two positive solutions if either $2N/(N+2) < p < 3$ and $1 < q < p$, or $p \geq 3$ and $p > q > p^* - 2/(p-1)$.

Introduction. We study the existence of a second positive solution for the problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{q-2} u + |u|^{p^*-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $1 < q < p < N$, $\lambda > 0$ and $p^* = Np/(N-p)$.

We will summarize the more relevant results related with this problem.

- (1) The problem without the critical term is much more easy. It was studied in detail in [GP1].
- (2) Any solution $u \in W_0^{1,p}(\Omega)$ of P_λ is regular ($u \in C^{1,\alpha}(\Omega)$). The main tool in the proof of this result is an extension of a lemma by Trudinger [TR], which allows us to prove that $u \in L^r$ with $r > p^*$, and a bootstrap argument. (See Appendix 1).
- (3) If λ is small enough, there exists a positive solution u_o , which is a local minimum of the energy functional. Moreover there exists infinitely many solutions of (P_λ) that possibly changes the sign. (See [GP2].)