

# *Singular Sets of Solutions to Elliptic Equations*

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ABSTRACT. We study the structure of the singular sets of solutions to elliptic differential equations in  $\mathbb{R}^n$ , the subsets where the solutions, with their gradients, vanish. We prove that such sets are countably  $(n - 2)$ -rectifiable under the assumption that the solutions vanish at finite order. We construct the tangent spaces for all points in the singular sets.

**1. Introduction.** Recently, there have been some papers studying the nodal sets of solutions to elliptic and parabolic equations. In a series of papers, Donnelly and Fefferman [DF1, DF2, DF3] studied the nodal sets of eigenfunctions of the Laplacian on a compact Riemannian manifold. They found both upper and lower optimal bounds of Hausdorff measures of nodal sets in the case that the given manifold is analytic. We refer to [DF3] and [D] for the case that the metric is smooth and the manifold is 2-dimensional. In [L], Lin obtained an optimal upper bound of the Hausdorff measures of nodal sets of solutions for second order elliptic equations with analytic coefficients and parabolic equations with time independent analytic coefficients. On the other hand, Hardt and Simon [HS] studied nodal sets of solutions to a general class of elliptic equations of second order without analyticity hypotheses. They also found an explicit estimate, though it may not be optimal, on local Hausdorff measures of nodal sets under a natural growth assumption on solutions. In [HL], we generalized the results in [HS] to second order parabolic equations without the analyticity.

Methods used by these authors are quite different. But, all their arguments depend, to some extent, on the structure of the singular sets, the sets where solutions and their gradients both vanish.

For the surface case, the singular sets of eigenfunctions consist of isolated points. In fact, the number of these points, including multiplicity, can be estimated. For details, see [D]. Generally, the singular sets of solutions could be very complicated. In [HS], Hardt and Simon proved that for a classical solution with relatively high order derivatives the singular set is a countable union of subsets of correspondingly smooth  $(n - 2)$ -dimensional submanifolds. Thus, it is