

# Two Results on Separating Vectors

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ABSTRACT. We show that a linear space of bounded operators with denumerable Hamel basis has a separating vector if and only if its subspace of finite rank operators has a separating vector, and also that the weakly closed algebra generated by a triangular operator has a separating vector. Moreover, we prove density of the set of all separating vectors.

A *separating vector* for a linear subspace  $\mathcal{S}$  of linear transformations on a vector space  $V$  is a vector  $x \in V$  for which the map  $S \rightarrow Sx$  from  $\mathcal{S}$  to  $V$  is injective. The purpose of this note is to settle two separating vector problems which arose in our earlier work and which have been open for several years.

In Theorem 8 we answer a question in [L], which arose in a study of reflexivity, and show that if  $\mathcal{S}$  is a linear subspace of  $\mathcal{B}(X)$  for a Banach space  $X$ , and if  $\mathcal{S}$  has a denumerable Hamel basis, then  $\mathcal{S}$  has a separating vector if and only if the linear subspace  $\mathcal{S}_F$  of finite rank operators in  $\mathcal{S}$  has a separating vector. Moreover, in this case  $\text{sep}(\mathcal{S})$  is dense in  $X$ , where  $\text{sep}(\mathcal{S})$  denotes the set of *all* separating vectors for  $\mathcal{S}$ . In the types of applications we have in mind,  $X$  will often be an operator algebra.

In Corollary 12 we show that if  $T$  is a Hilbert space operator which is triangular in the sense that there is an orthonormal basis  $\{e_i\}$ , such that  $Te_n \in \text{span}\{e_1, \dots, e_n\}$  for each  $n$ , then the weak operator topology closed algebra  $\mathcal{W}(T)$  generated by  $T$  and  $I$  has a separating vector. (In fact, we prove the stronger result that  $\text{sep}(\mathcal{W}(T))$  is dense, and we obtain this in a somewhat wider setting.) The question of validity of this was raised formally in [LW1] and [HLW], although it had been considered earlier, and was motivated by a counterexample constructed in [W] of a Hilbert space operator  $S$  for which  $\mathcal{W}(S)$  fails to have a separating vector. Theorems 8 and 11 are related in that the ideas that led to a proof of the first inspired the proof of the second.

We refer the reader to [H] for an extensive exposition of triangular operators, and to [LW1] for a discussion of some open questions. In this article all vector spaces will be assumed complex. If  $V$  is a vector space we use  $\mathcal{L}(V)$  to denote