

An Extension of Szegő's Theorem

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ABSTRACT. In this paper we examine the extent to which the divergence of an integral of the type given by Szegő's Theorem implies the existence of a nontrivial $L^t(\mu)$ -summand in $P^t(\mu)$ (the closure of the polynomials in $L^t(\mu)$) for measures μ whose supports are topologically equivalent to $\{z : |z| < 1 \text{ and } \text{Im}(z) \leq 0\} \cup \{z : |z| = 1\}$. We also show that, in contrast with the setting of Szegő's Theorem, outer functions can have only a limited role in our investigation.

1. Introduction. A natural starting point for us is a general form of Szegő's Theorem (see [G; Theorem 3.1, p. 144], or Theorem 1.1 below), which requires a brief preliminary discussion of harmonic measure.

For any bounded, simply connected region G in the complex plane \mathbf{C} , and for any point z_0 in G , we let $\omega(\cdot, G, z_0)$ (or just ω_G) denote harmonic measure on ∂G for the (norming) point z_0 . Indeed, $\omega(\cdot, G, z_0)$ is the unique probability measure, given by the Riesz Representation Theorem, with the property that for any continuous real-valued function h on ∂G , $\int_{\partial G} h(\xi) d\omega(\xi, G, z_0) = \hat{h}(z_0)$ —the evaluation at z_0 of the harmonic function \hat{h} on G that has boundary values h . By Harnack's Inequality, $\omega(\cdot, G, z_1)$ and $\omega(\cdot, G, z_2)$ are boundedly equivalent for any two points z_1 and z_2 in G . For this reason, most of the results of this paper are independent of the choice of the norming point in G , and so we predominantly use the ω_G notation for harmonic measure.

We are now in a position to state a general form of Szegő's Theorem. In the statement of this theorem, and throughout the rest of this paper, we let \mathbf{D} denote the unit disk $\{z : |z| < 1\}$ and let P denote the collection of polynomials.

Theorem 1.1 (G. Szegő). *Let Ω be a Jordan domain, z_0 be a point in Ω and ψ be a conformal map from Ω onto \mathbf{D} such that $\psi(z_0) = 0$. If μ is a finite, positive Borel measure with support in $\partial\Omega$ and $\mu = \mu_a + \mu_s$ is the Lebesgue*