

The Phragmén-Lindelöf Theorem for Elliptic Partial Differential Equations

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We consider linear elliptic partial differential equations of the form

$$(1) \quad L(u) \equiv au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0, \quad (a > 0).$$

It is known that these equations obey the maximum principle, namely that a non-constant solution cannot take on a maximum or minimum in the interior of its domain of definition. For harmonic functions a strengthening of the usual maximum principle is provided by the classical Phragmén-Lindelöf theorem. We state the analogue for equation (1) in the following rough form: if a solution of (1) defined in a region lying in a half plane is non-negative on the boundary and is $o(r)$ as $r \rightarrow \infty$, it must be non-negative in the entire region. In the present note this result is proved for solutions of (1) under some mild assumptions concerning the coefficients. The theorem remains true if $L(u)$ is replaced by $L(u) + \lambda u$, where $\lambda(x, y) \leq 0$. The method is patterned after a well known proof of the Phragmén-Lindelöf theorem based on the harmonic measure as majorant.¹ The essential idea of our proof is to exploit the fact that the majorant need not be chosen among solutions of (1) but rather among the wider class of functions obeying only the inequality $L(u) \leq 0$. This is an attack made possible by Hopf's form of the maximum principle,² which asserts that a non-constant function satisfying the inequality $L(u) \leq 0$ cannot take on an interior minimum. Among the functions available as majorant it suffices to find one with certain growth properties possessed in particular by the harmonic majorant of the classical

¹ See R. NEVANLINNA, *Eindeutige Analytische Funktionen*, Springer, Berlin 1936, p. 43.

² E. HOPF, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte Akad. Berlin 1927, 147-152; see also BATEMAN, *Partial Differential Equations*, Cambridge 1932, Dover Publ. 1944, p. 135-137.