

# *Remarks on the Preceding Paper by D. Gilbarg*

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In the preceding paper D. Gilbarg communicated an important generalization of the Phragmén-Lindelöf theorem to linear elliptic differential equations of second order and in two independent variables. It is our object to prove this theorem for  $n$  independent variables. Suppose, as Gilbarg does, that the coefficients of the differential operator

$$(1) \quad L(u) = a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + b_i(x) \frac{\partial u}{\partial x_i}$$

are continuous in the half space  $x_n > 0$ ,  $x = (x_1, \dots, x_n)$ , and that  $L$  is uniformly elliptic in  $x_n > 0$ :

$$(2) \quad \frac{\text{Max}_{i,k} |a_{ik}(x)|}{\lambda(x)} < A$$

is bounded in  $x_n > 0$  if  $\lambda(x) > 0$  denotes the smallest proper value of the quadratic form  $a_{ik}(x)\xi_i\xi_k$ . Uniform ellipticity means, equivalently, that

$$\frac{\Lambda(x)}{\lambda(x)}$$

is bounded in  $x_n > 0$  if  $\Lambda(x)$  is the greatest proper value.

**Theorem.** Suppose that the coefficients  $a_{ik}(x)$  and  $b_i(x)$  are continuous and that  $L$  is uniformly elliptic in  $x_n > 0$ . Suppose moreover that there is a decreasing function  $p(r) > 0$  such that

$$(3) \quad \int_0^\infty p(r) dr < \infty, \quad \sum_i \frac{|b_i(x)|}{\lambda(x)} < p(r), \quad r^2 = x_i x_i.$$