

# *Bounds for Solutions of Laplace's Equation Satisfying Mixed Conditions*

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In this paper upper and lower bounds are established for the integrals of the Laplace equation which satisfy mixed conditions on the frontier of the open set considered. These bounds are obtained by introducing two auxiliary functions whose existence is assumed. Two suitable auxiliary functions are then exhibited for a particular boundary value problem.

**a.** Let us indicate by  $x$  a general point of a Euclidean space  $S_r$  of  $r$  dimensions, with coordinates  $(x_1, x_2, \dots, x_r)$ . Let  $A$  be an open set of  $S_r$  and  $\mathfrak{F}A$  its boundary, and further let  $f(x)$  be a real function in  $A$  and  $\alpha(x)$ ,  $\beta(x)$ ,  $\gamma(x)$  be three real functions defined in  $\mathfrak{F}A$ . Let us further suppose  $|\alpha(x)| + |\beta(x)| > 0$  for  $x \in \mathfrak{F}A$ , and that the set  $H$  of the points of  $\mathfrak{F}A$  for which  $\beta(x) \neq 0$  is not empty, and also that through every point  $x$  of  $\mathfrak{F}A$  for which  $\alpha(x) \neq 0$  there is defined an axis  $l$  which penetrates into  $A$ .<sup>1</sup>

Let us write  $\Gamma$  for the class of functions continuous in  $A + \mathfrak{F}A$ , having first and second partial derivatives continuous in  $A$ , and having for every point  $x$  of  $\mathfrak{F}A$  for which  $\alpha(x) \neq 0$  a directional derivative, with respect to the axis  $l$  which is defined at such a point. With this as a premiss, we have the following theorem:

**Theorem:** *Let the function  $u(x)$ , belonging to the class  $\Gamma$ , be an integral in  $A$  of the Laplace equation*

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<sup>1</sup> An axis  $l$  which originates at a point  $x$  on the boundary  $\mathfrak{F}A$  is said to penetrate into  $A$  if there exists on it a point  $y$  such that the interior points of the segment  $xy$  belong to  $A$  and the sense of the vector  $xy$  is the same as that of the axis  $l$ .