

*On the Phragmén-Lindelöf Principle for Elliptic Differential Equations**

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This paper supplements and extends the conclusions of Gilbarg [1] and Hopf [2] concerning the Phragmén-Lindelöf theorem for linear elliptic partial differential equations. We consider especially the equation

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0,$$

where the domain of definition of the coefficients is the upper half plane $y > 0$. We suppose that L is *uniformly elliptic*; that is, there exists a positive constant k such that at every point $z = (x, y)$ in the upper half plane and for all real λ, μ ,

$$(1) \quad \frac{a\lambda^2 + 2b\lambda\mu + c\mu^2}{a + c} \geq k(\lambda^2 + \mu^2).$$

Moreover d and e are to tend to zero as $r \rightarrow \infty$, $r^2 = x^2 + y^2$, in such a way that there is a decreasing function $p(r)$ satisfying

$$(2) \quad \int_0^\infty p(r) \, dr < \infty, \quad \frac{(d^2 + e^2)^{\frac{1}{2}}}{a + c} \leq p(r).$$

For the purposes of this paper it will not be necessary to make any further assumptions regarding the coefficients; for example, they may be discontinuous or unbounded. The Gilbarg-Hopf theorem, which is an analogue for elliptic equations of the classical Phragmén-Lindelöf theorem for harmonic functions, states that if a function $u = u(z)$ satisfies the inequality $L(u) \geq 0$ in some sub-

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