

# On the Singularities of Zonal Harmonic Expansions

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## 1. INTRODUCTION

**1.1.** The purpose of this note is to prove the following

**Theorem:** Let

$$(1) \quad a_0 + a_1 r P_1(\cos \theta) + a_2 r^2 P_2(\cos \theta) + \cdots + a_n r^n P_n(\cos \theta) + \cdots$$

be an expansion in terms of zonal harmonics with real coefficients  $a_n$  and

$$(2) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1.$$

We consider the power series with the same coefficients:

$$(3) \quad a_0 + a_1 \zeta + a_2 \zeta^2 + \cdots + a_n \zeta^n + \cdots .$$

Let  $u(r, \theta)$  be the zonal harmonic function defined by (1), and  $f(\zeta) = f(re^{i\theta})$  the analytic function defined by (3). Both functions are regular for  $r = |\zeta| < 1$ ; moreover, the boundary point  $(1, \theta)$  is a regular (singular) point of  $u(r, \theta)$  if and only if the boundary point  $\zeta = e^{i\theta}$  is a regular (singular) point of  $f(\zeta)$ .

As usual, the symbol  $P_n$  stands for the  $n^{\text{th}}$  Legendre polynomial,  $\theta$  being the spherical distance from the pole. That the coefficients be real is not essential.

**1.2.** In spite of its simplicity, this theorem does not seem to be found in the literature. Its significance is obvious: by its aid, practically all theorems on power series concerning the regular or singular character of the points on the circle of convergence lead to analogous assertions for zonal harmonic expansions. As an interesting example we cite the well known theorem of Fabry on gap series, which yields the following result.

Concerning the series (1) we make the same assumptions as above. Moreover let