

# *Inequalities for a Classical Eigenvalue Problem\**

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**§1. Introduction.** (a) Let  $\mathfrak{D}$  be a simply-connected domain in the  $xy$ -plane bounded by the simple closed curve  $\mathfrak{C}$ , assumed to be analytic. We consider the problem of determining those twice-continuously differentiable functions  $\phi = \phi(x, y)$  which satisfy

$$(1.1) \quad \nabla^2 \phi = 0 \quad \text{in } \mathfrak{D},$$

$$(1.2) \quad \frac{\partial \phi}{\partial n} = h\phi \quad \text{on } \mathfrak{C},$$

where  $\partial/\partial n$  indicates differentiation with respect to the *exterior* normal to  $\mathfrak{C}$ ,  $\nabla^2$  is the two-dimensional laplacian operator, and  $h$  is a constant. We call this the “problem of Stekloff” [5].

It can be shown [1, 2] that only for a discrete infinite set of non-negative real values of  $h$  (eigenvalues) do there exist functions  $\phi$  (eigenfunctions) that satisfy (1.1) and (1.2). We denote the totality of eigenvalues by  $h_0, h_1, h_2, \dots$  (with  $h_k \leq h_{k+1}$  for all  $k$ ), and the corresponding eigenfunctions by  $\phi_0, \phi_1, \phi_2, \dots$ . We suppose the latter to be normalized according to the rule

$$(1.3) \quad \int_{\mathfrak{C}} \phi_k^2 ds = 1 \quad (k = 0, 1, 2, \dots).$$

(It is clear from (1.1) and (1.2) that  $h_0 = 0, \phi_0 = \text{constant}$ .) The main interest of the present paper lies in determining an upper bound for the first non-trivial

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