

# *The Boundary Behavior of a Quasi-Conformal Mapping*

A. J. LOHWATER

*Communicated by L. BERS*

It is a well-known result of BEURLING [3] in the theory of conformal mapping that if  $w = f(z)$  is analytic and univalent in  $|z| < 1$  then the radial limits  $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$  exist for all points  $e^{i\theta}$  on  $|z| = 1$  except for a possible set of logarithmic capacity zero; furthermore, the set of points  $e^{i\theta}$  on  $|z| = 1$  on which  $f(e^{i\theta})$  is constant cannot have positive capacity. In this note, we shall show that the theorem of BEURLING can be extended to the class of functions which give a quasi-conformal mapping of  $|z| < 1$  onto a simply-connected region of the  $w$ -plane. We remark that E. SAKAI [6], and later S. AGMON [1], have shown that the radial limit values of such a mapping exist for all  $e^{i\theta}$  of  $|z| = 1$  with the possible exception of a set of linear measure zero; in both papers, the existence of such boundary values has been shown only for *radial* approach. (This result was applied by L. BERS [2] to a problem in the theory of non-linear elliptic partial differential equations, and, while it is not known whether the present result will have application in that theory, the result appears to be of independent interest.) BEURLING's proof of his theorem for the conformal case does not appear capable of extension to the quasi-conformal case, but a proof of BEURLING's theorem which is due to TSUJI [7] provides the idea for an extension.<sup>1)</sup> We prove first (Theorem 1) a simple corollary of a theorem of NOSHIRO [4] on the boundary values of quasi-conformal mappings, which is needed in the proof of the main result (Theorem 2). In Theorem 3, the requirement that a quasi-conformal mapping be univalent is removed, provided that the function map  $|z| < 1$  onto a Riemann surface having a bounded spherical area.

---

<sup>1)</sup>The author has been informed recently by Professor BERS that J. A. JENKINS has made a similar extension of BEURLING's theorem.

*Editor's note:* see the following paper.