

# *Absolute Convergence and Integrability of Trigonometric Series*

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**1. Introduction.** Suppose that  $f(x)$  is defined on  $(0, \pi)$  by the absolutely convergent sine series  $\sum b_n \sin nx$ . (By this we mean that the series converges absolutely for all  $x$ , so that  $\sum |b_n|$  converges; similarly for other trigonometric series.) If we expand  $f(x)$  in a cosine series  $\sum a_n \cos nx$ , this cosine series will not necessarily be absolutely convergent (Theorem 10, below). It is therefore of interest to find sufficient conditions for the absolute convergence of the cosine series. A rather general one that we shall obtain is

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\frac{1}{2}n} \frac{|b_{n+k} - b_{n-k}|}{k} < \infty.$$

This implies some more special criteria, for example, that the cosine series converges absolutely if  $b_n$  is ultimately decreasing, or if  $\sum |b_n| \log n$  converges, or if  $\sum n |\Delta b_n|$  converges.

When  $f(x)$  is defined by an absolutely convergent sine series, the improper integral  $\int_{-\pi}^{\pi} x^{-1} f(x) dx$  always converges [1]. We shall show that when the cosine series also converges absolutely,  $|x^{-1} f(x)|$  is necessarily integrable. Thus any sufficient condition for the absolute convergence of the cosine series of a function  $f(x)$  defined by an absolutely convergent sine series is in particular a sufficient condition for the integrability of  $|x^{-1} f(x)|$ . A number of known theorems, and some new ones, thus follow from our criteria. (However, these theorems can be proved much more easily directly.) By partial summation, a condition on  $b_n$  that makes  $|x^{-1} f(x)|$  integrable is equivalent to a condition on  $\Delta a_n$  that makes  $\sum a_n \cos nx$  a Fourier series (*i.e.*, the Fourier series of a Lebesgue integrable function). Thus we also obtain a number of theorems on the Fourier character of a cosine series. The most general of these is that  $\sum a_n \cos nx$  is a Fourier