

# On a Complementary Energy Principle

D. TRIFAN

Communicated by HILDA GEIRINGER

In the mathematical theory of elasticity, two important principles have been widely used, namely the Theorem of Minimum Potential Energy, a minimal principle for strains, and Castigliano's Principle, a minimal principle for stresses occasionally referred to as a complementary energy principle [1]. On the other hand, numerous minimal principles have been developed in the mathematical theory of plasticity due to the need of numerous stress-strain relations. We shall consider here a stress-strain relation for homogeneous, isotropic, incompressible, strain hardening materials exhibiting a gradual transition from the elastic to the plastic state, the so-called stress theory of plastic flow [2]. In differential form the stress-strain relations for loading and unloading respectively are

$$ds_{ij} = 2G_0 de_{ij} - p(E)e_{ij} dE \quad \text{for } dE > 0 \quad (1a)$$

$$ds_{ij} = 2G_0 de_{ij} \quad \text{for } dE < 0 \quad (1b)$$

where the differential of stress deviation  $ds_{ij}$  and the differential of strain deviation  $de_{ij}$  are defined in terms of the differentials of stress tensor  $d\sigma_{ij}$  and strain tensor  $d\epsilon_{ij}$  by  $ds_{ij} = d\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$  and  $de_{ij} = d\epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}$ —repeated indices indicate summation in accordance with the summation convention of tensor calculus, and  $\delta_{ij}$  is the Kronecker delta.  $G_0$  is the shear modulus in the elastic range,  $p(E)$  is a positive definite function, dependent on the material under consideration, of the strain invariant  $E$  defined as  $E = \frac{2}{3}e_{ij}e_{ij}$ , and  $dE$  is the loading-unloading criterion. For incompressible materials  $\epsilon_{kk} = 0$  so that  $e_{ij} = \epsilon_{ij}$ . By means of stress-strain relations (1) a minimal principle for the differential of strain deviation has been developed [3]. The purpose of this paper is to introduce a minimal principle for the differential of stress deviation, *i.e.* a complementary energy principle. This pair of minimal principles for the *stress theory of plastic flow* can be regarded as analogues to the HODGE-PRAGER Principle [4] and the accompanying one established by GREENBERG [5] for the so-called *strain theory of plastic flow*.

The stress-strain relations (1) have the differential of stress deviation as a function of the strain deviation and the differential of strain deviation. For our purposes inverted forms must be obtained so that the differential of strain