

A Criterion for Homogeneous Linear Differential Equations with Damped Solutions

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1. The results of this paper center about the following Abelian lemma:

(i) *If $\omega = \omega(r)$, where $0 < r \leq r_0$, is any continuous function for which the product $r\omega(r)$ stays between two positive bounds as $r \rightarrow 0$, then there exists a constant $\alpha = \alpha(\omega) > 0$ having the property that*

$$(1) \quad x(r) = O(r^\alpha) \quad \text{as } r \rightarrow 0$$

(and so, in particular, $x(+0) = 0$) holds for every solution $x(r)$ of the differential equation

$$(2) \quad \frac{d^2x}{dr^2} + \omega^2x = 0.$$

If in addition $\liminf r\omega(r) > \frac{1}{2}$, then, on the one hand, (1) holds for $\alpha = \frac{1}{2}$ (no matter what the value of $\limsup r\omega(r) < \infty$ may be) and, on the other hand, (2) is stable in the sense of being oscillatory (that is, every real-valued solution $x(r) \equiv 0$ of (2) must change sign an infinity of times as $r \rightarrow 0$).

The last assertion (but nothing like the estimate (1) or, for that matter, just $x(+0) = 0$) is clear from STURM'S comparison theorem. For, according to that theorem, it is sufficient to ascertain the oscillatory character of (1) when $r\omega = r\omega(r)$ is a constant $\omega_0 > \frac{1}{2}$. But a direct substitution shows that, if $r\omega(r) = \omega_0$, then $x(r) = r^\lambda$ is a solution of (2) if $\lambda^2 - \lambda + \omega_0^2 = 0$. Since the roots of this quadratic equation are $\lambda = \frac{1}{2} \pm i\beta$, where $\beta > 0$, if $\omega_0 > \frac{1}{2}$, the assertion follows from the fact that the zeros of the cosine or of the sine of $\beta \log r$ cluster at $r = +0$.

A (formal) logarithmic differentiation of the O -estimate of (i), or of the corresponding O -estimate in (ii) below, shows that the issues involved are related to POINCARÉ'S problem concerning RICCATI'S equation (*cf.*, *e.g.*, [2], pp. 53–54). Correspondingly, the O -assertions in question are of the same type as in classical considerations of LIAPOUNOFF [6], pp. 223–224, and PERRON [7], [8]. But there the assumptions are quite different and, mainly because of the inclusion of the oscillatory case, quite a different approach will have to be