

Debye Potentials

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Communicated by A. ERDÉLYI

1. Introduction. It is well known that if u and v are functions that satisfy the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

in a domain D and if \mathbf{r} is a position vector relative to an origin lying outside D then the vector fields

(1.1) $\mathbf{A} = \nabla \times (\nabla \times \mathbf{ur}) + k\nabla \times \mathbf{vr}$, $\mathbf{B} = \nabla \times (\nabla \times \mathbf{vr}) + k\nabla \times \mathbf{ur}$,
satisfy Maxwell's equations

$$\nabla \times \mathbf{A} = k\mathbf{B}, \quad \nabla \times \mathbf{B} = k\mathbf{A},$$

for a time-harmonic electromagnetic field in D . The functions u and v were introduced into electromagnetic theory by P. DEBYE in 1909 [10] and are called Debye potentials for the electromagnetic field (\mathbf{A}, \mathbf{B}) . They are applied frequently in solving boundary value problems for Maxwell's equations [6, 19]. This is done, typically, by assuming that the unknown electromagnetic field has a representation by Debye potentials, using the conditions of the problem to determine what u and v must be, and then verifying that the fields \mathbf{A} and \mathbf{B} generated by this pair of potentials actually solve the original problem. This indirect procedure is designed to avoid the question of how general an electromagnetic field can be represented by Debye potentials. The present paper provides an answer to this question by showing that every electromagnetic field defined in a region between two concentric spheres can be represented there by Debye potentials. The theorem is shown to follow from a very general decomposition theorem for vector fields defined on a sphere (the Hodge decomposition theorem). Two applications of the theorem are given. They are, first, a proof that every electromagnetic field defined in a spherical domain can be expanded in a series of electromagnetic multipoles and, second, several theorems on the behavior of electromagnetic fields near isolated singular points.

The electromagnetic multipoles are the fields

$$\mathbf{A}_n = \nabla \times (\nabla \times u_n \mathbf{r}), \quad \mathbf{B}_n = k\nabla \times u_n \mathbf{r},$$