

Asymptotic Forms for the Whittaker Functions with Both Parameters Large

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1. Introduction. Certain solutions of the differential equation

$$(1.1) \quad \frac{d^2 W}{dx^2} + \left\{ -\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2} \right\} W = 0$$

are called Whittaker functions and are customarily designated $W_{k,m}(x)$ and $M_{k,m}(x)$. Their asymptotic behavior is different according as one, two, or three of the quantities $|x|$, $|k|$, and $|m|$ are large. Recent books by H. BUCHHOLZ [2] and F. TRICOMI [10] contain summaries of many investigations in this area. Other results are to be found in [3, 4, 5, 6]. An examination of this literature reveals that least is known of the structure of the Whittaker functions if both k and m are unbounded. When $|x|$ is also large, such information as is available applies only to nearly real values of x and k and nearly real or pure imaginary values of m and is subject to the further limitation that the ratios x/k and x/m be nearly fixed [3, 4, 5].

In this paper the asymptotic behavior of the Whittaker functions with x , k , and m real or complex is obtained for both $|k|$ and $|m|$ large and x unrestricted under the hypothesis that $(m^2 - k^2)/k$ be bounded, so that $\lim_{|k| \rightarrow \infty} m/k = \pm 1$. The results are stated in Theorems 1, 2, 4, and 5 of the sequel. The analysis to be given is based upon the methods of asymptotic solution of ordinary differential equations containing a large parameter developed by R. E. LANGER. The author hopes to investigate the behavior of the Whittaker functions for other configurations of large k and m subsequently, using these same methods. In this paper particular reference is made to work of R. MCKELVEY [9]. The saddle-point and power series methods have been the principal tools in previous investigations [3, 4, 5, 10]. It should be remarked that in the notation of the confluent

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