

A Markovian Decision Process

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1. Introduction. The purpose of this paper is to discuss the asymptotic behavior of the sequence $\{f_N(i)\}$, $i = 1, 2, \dots, M$, $N = 1, 2, \dots$, generated by the non-linear recurrence relations

$$(1) \quad f_N(i) = \text{Max}_{\mathbf{q}} \left[b_i(\mathbf{q}) + \sum_{j=1}^M a_{ij}(\mathbf{q}) f_{N-1}(j) \right], \quad N = 1, 2, \dots, \\ f_0(i) = c_i, \quad i = 1, 2, \dots, M.$$

Although these equations are non-linear, they possess certain quasi-linear properties which permit a more thorough discussion than might be imagined upon first glance.

As we shall discuss below, this question arises from the consideration of a dynamic programming process. A related process gave rise to an equation of the above form which was discussed in [1], under a particular set of assumptions concerning the functions $b_i(\mathbf{q})$ and the matrices $A(\mathbf{q}) = (a_{ij}(\mathbf{q}))$. Here we shall impose restrictions of a quite different type. Any complete discussion of relations of the foregoing type is at least as detailed as a corresponding discussion of the linear case, and, as in the linear case, the assumptions made determine the techniques employed to a considerable extent.

We shall discuss elsewhere some interesting quadratically non-linear recurrence relations which arise from specializing the forms of $b_i(\mathbf{q})$ and $a_{ij}(\mathbf{q})$. These are related to the types of differential equations discussed in [3], being essentially a particular type of discrete version.

It will be clear from what follows that similar techniques can be utilized to treat the determination of the asymptotic behavior of the sequences defined by

$$(2) \quad f_N(x) = \text{Max}_{\mathbf{q}} \left[b(x, \mathbf{q}) + \int_0^1 K(x, y, \mathbf{q}) f_{N-1}(y) dy \right], \quad N = 1, 2, \dots, \\ f_0(x) = c(x), \quad 0 \leq x \leq 1,$$

and by the equation

$$(3) \quad u_N = \text{Max}_{\mathbf{q}} \left[b(\mathbf{q}) + \sum_{i=1}^k a_i(\mathbf{q}) u_{N-i} \right], \quad N = k + 1, k + 2, \dots, \\ u_i = c_i, \quad i = 0, 1, \dots, k,$$

under corresponding assumptions.