

*On the Change of Natural Frequencies Induced by Small Constraints**

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1. Introduction. When a constraint is added to a vibrating linear mechanical system the frequency of each normal mode tends to increase. It is the aim of this paper to show how this qualitative statement can be expressed in a quantitative form. A short outline of the applications of the theory presented here appeared in an earlier paper of the authors [1].

Consider a vibrating linear system with constraints which can be varied continuously. We wish to find the small increase in frequency resulting from a small increase of constraint. The original system will be called the *unconstrained system* S ; the system with the small additional constraints, the *constrained system* S' .

Our problem is resolved by the following simple formula

$$(A) \qquad \delta\lambda = \delta V.$$

Here λ is the square of the circular frequency of a given normal mode of vibration of S , and $\delta\lambda = \lambda' - \lambda$ is the increase in λ resulting from the increase in constraints. To maintain this increase in constraint, external forces of amplitude F must be applied to S ; the resulting vibration of S' is assumed to be suitably normalized. Now let the unconstrained system S be at rest. Let the forces F act to produce a static deflection. Then δV is the potential energy of S in this state of static deflection (Section 4).

In many cases the normalized mode of vibration is known for the constrained system S' . The forces F can then be deduced. Hence $\delta\lambda$ can be obtained without knowing any of the other constrained or unconstrained modes.

Formula (A) holds in the simple form given here when the eigenvalue λ' is non-degenerate. In the case of degenerate states the frequency changes are obtained by solving a secular equation (Section 5). Formula (A) is also valid with a dual interpretation of the potential energy δV in terms of the displace-

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