

On Jordan Algebras with Two Generators

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1. Introduction. A (non-associative) algebra \mathfrak{A} is called a Jordan algebra if its multiplication satisfies the identities

$$(1.1) \quad ab = ba, \quad (a^2b)a = a^2(ba).$$

We shall assume that the base ring Φ or \mathfrak{A} is a field of characteristic not two. If \mathfrak{A} is an associative algebra over a field of characteristic not two and multiplication composition (\times) , then \mathfrak{A} defines a Jordan algebra \mathfrak{A}^+ relative to the Jordan multiplication $ab = \frac{1}{2}(a \times b + b \times a)$. A Jordan algebra is called special if it is isomorphic to a subalgebra of an algebra \mathfrak{A}^+ , \mathfrak{A} associative. It has been known for a long time that there exist exceptional (non-special) Jordan algebras and it has been shown by P. COHN [2] that there exist special Jordan algebras with homomorphic images which are not special.

Recently it was proved by A. I. SHIRSHOV [7] that the free Jordan algebra with two generators is special. The main purpose of this note is to give a comparatively simple alternative proof of this result which gives at the same time a simple basis for the algebra.² A sketch of the procedure is as follows. We begin by studying the free special Jordan algebra \mathfrak{J}_0 with two generators. This is the subalgebra relative to Jordan multiplication generated by two generators in a free associative algebra \mathfrak{F} . We obtain a basis for \mathfrak{J}_0 which is defined inductively using Jordan multiplication (§3). We then imitate this procedure in the free Jordan algebra \mathfrak{J} , thus defining a set S in \mathfrak{J} in 1-1 correspondence with the basis for \mathfrak{J}_0 . To prove that S is a basis it suffices to show that $[S]^2 \subset [S]$, where $[S]$ is the space spanned by S . The proof of this makes use of some three variable identities which are of independent interest (§2).

In considering \mathfrak{J}_0 , it is possible to encompass the case of characteristic two by defining a special Jordan algebra as a vector space with a unary composition $a \rightarrow a^{[2]}$ and a binary composition $(a, b) \rightarrow \{aba\}$ such that there is a 1-1 linear

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²SHIRSHOV's result is valid also for rings containing no elements of additive order two. Our argument could be extended to that case but we shall not insist on this.