

Dynamic Programming and the Variation of Green's Functions

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1. Introduction. In an earlier paper [1], the functional equation technique of dynamic programming was applied to obtain a variational equation for a Green's function corresponding to a second order ordinary differential equation. In the present paper, this method is extended to apply to elliptic partial differential operators, and a first consequence is the classical Hadamard variational formula. Further results require a more detailed argument which we shall present subsequently.

The technique presented here utilizes the principle of optimality (see [2]) in the following fashion. Given a one-parameter family of regions, monotone under inclusion, one takes the minimum value of a certain integral on any given region R , subject to certain restrictions, to be a functional of those restrictions and the region R . Then if $R^* \subset R$ the functional on R can be approximated by means of a related functional on R^* satisfying slightly different restrictions. This leads to a Gâteaux difference equation from which one easily derives the Hadamard relation.

The method is initially presented for the Laplace operator on R , and appropriate generalizations are indicated in §6 and §7.

2. Preliminaries. Let R be a bounded connected region of n -dimensional real Euclidean space, whose boundary ∂R is of class C_2 . For convenience we shall not explicitly write out the differentials of volume and surface area in integrals over R and ∂R . Given any twice-differentiable function u on R , let Δu and u_p represent the Laplacian of u and the restriction of u to its limiting values on ∂R , respectively. Then, if v and w are suitable functions on R and ∂R , the boundary value problem

$$(1) \quad \Delta u = v, \quad u_p = w$$

possesses the unique solution

$$(2) \quad u(p) = \int_{q \in R} g(p, q)v(q) + \int_{q \in \partial R} g_n(p, q)w(q),$$