

# Representation of Functions by Convolutions

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**1. Introduction.** To state the problem with which we are concerned in this paper in its simplest form, consider the class  $L^1(\mathbb{R}^1)$ , *i.e.*, the set of all complex-valued functions on the real line  $\mathbb{R}^1$  which are integrable on  $\mathbb{R}^1$  in the sense of Lebesgue. If  $g$  and  $h$  belong to  $L^1(\mathbb{R}^1)$ , it is well known that the integral

$$(1.1) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)h(t) dt$$

exists for almost all  $x \in \mathbb{R}^1$  and represents a member  $f$  of  $L^1(\mathbb{R}^1)$  which is called the convolution of  $g$  and  $h$ ; in symbols

$$(1.2) \quad f = g * h.$$

The Fourier transform of  $f \in L^1(\mathbb{R}^1)$  will be denoted by  $\hat{f}$ :

$$(1.3) \quad \hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iyx} dx \quad (y \in \mathbb{R}^1).$$

The relation (1.2) is then equivalent to

$$(1.4) \quad \hat{f}(y) = \hat{g}(y)\hat{h}(y) \quad (y \in \mathbb{R}^1).$$

The problem under consideration is the following: Which functions  $f \in L^1(\mathbb{R}^1)$  can be represented in the form (1.2) (for almost all  $x$ ) with  $g$  and  $h$  in  $L^1(\mathbb{R}^1)$ ? Equivalently, which Fourier transforms are products of Fourier transforms, as in (1.4)? The answer is that *every function in  $L^1(\mathbb{R}^1)$  is a convolution*. That is to say, every element of the convolution algebra  $L^1(\mathbb{R}^1)$  can be factored in  $L^1(\mathbb{R}^1)$ , although this algebra lacks a unit. There are no primes in  $L^1(\mathbb{R}^1)$ .

This problem can of course be considered in a wider setting. If  $G$  is any locally compact abelian group (we write the group operation as addition) and  $\hat{G}$  is the group of all continuous characters of  $G$ , there is a translation invariant measure (Haar measure) on  $G$ , and we denote by  $L^1(G)$  the set of all complex functions on  $G$  which are integrable with respect to this measure. Convolution

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