

New Inequalities for Entire Functions

R. P. BOAS, Jr. & A. C. SCHAEFFER¹

Communicated by C. LOEWNER

1. Introduction. Let $f(z)$ be regular for $x \geq 0$ and of exponential type $\tau < \pi$. If $\lim_{n \rightarrow \infty} f(n)$ exists, it was shown by CARTWRIGHT that $\lim_{x \rightarrow \infty} f(x)$ exists, and by PLANCHEREL & PÓLYA, in analogy to S. BERNSTEIN's theorem on derivatives, that $\lim_{x \rightarrow \infty} f'(x) = 0$. The converse of the first statement is trivially true, while the converse of the second is false (consider, for example, $f(z) = z \int_0^\pi t^{-\frac{1}{2}} \sin zt \, dt$). One of our objects is to obtain generalizations of the second statement which have valid converses. We shall show, in particular, that $\lim_{n \rightarrow \infty} f(n)$ exists if and only if $\sum_1^\infty f'(n)$ converges. This is plausible since $\lim f(n)$ exists if and only if $\lim f(x)$ exists, *i.e.* if and only if $\int_0^\infty f'(t) \, dt$ converges. We deduce from this, answering a question of GAIER, a (C, 1) analogue of CARTWRIGHT's theorem: $f(n)$ approaches a limit in the (C, 1) sense if and only if $f(x)$ approaches a limit in the (C, 1) sense. That is, $m^{-1} \sum_1^m f(k)$ has a limit as $m \rightarrow \infty$ if and only if $x^{-1} \int_0^x f(t) \, dt$ has a limit as $x \rightarrow \infty$. More generally, if one of $m^{-1} \sum_1^m f(k + \lambda)$ and $x^{-1} \int_0^x f(t + \lambda) \, dt$ has a limit for some λ , the other has a limit for every positive λ . From this, in turn, we deduce that $f(n)$ approaches a limit (C, 1) if and only if $\sum f'(n)$ is summable (C, 1); that is, $\lim_{m \rightarrow \infty} m^{-1} \sum_0^m f(k)$ exists if and only if $\lim_{m \rightarrow \infty} m^{-1} \sum_0^m (m - k) f'(k)$ exists. In particular, if $f(x)$ approaches a limit (C, 1), then $f'(x)$ approaches zero (C, 1).

There are corresponding results with boundedness replacing convergence, and more precise results for entire functions. We now state our results in more detail.

Let $f(z)$ be an entire function of exponential type τ (that is, of order 1 and type τ or less, or of order less than 1). If $|f(x)| \leq M$ for all real x then, according to a famous theorem of S. BERNSTEIN, $|f'(x)| \leq M\tau$. The converse is, of course, not true; we cannot even infer the boundedness of $f(x)$ from that of $f'(x)$. Our first theorem is a generalization of BERNSTEIN's theorem that has a valid converse in the following sense: If $\tau < \pi$, $f(x)$ is bounded if and only if all sums $\sum_p^q f'(x + k)$ have a common bound (independent of p and q). More precisely, we prove

¹ This represents work done by Boas with partial support of the National Science Foundation and by SCHAEFFER under Contract N7 onr-28507 with the Office of Naval Research.