

# On the Cohomology Ring of a Sphere Bundle<sup>1</sup>

W. S. MASSEY

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**1. Introduction.** This paper is concerned with the general problem of determining the cohomology ring of a fibre space whose fibre is a sphere in terms of the cohomology of the base space and various invariants of the fibre space structure, such as characteristic cohomology classes (in particular, Stiefel-Whitney classes and Pontrjagin classes).

The first part of the paper is concerned with two formulas which give explicit relations between cup products in the total space of an orientable sphere bundle and triple products<sup>2</sup> in the base space (for which one of the arguments is the characteristic class of the bundle). In general, these formulas do not suffice to completely determine the cohomology ring of the total space. An example is given of a non-trivial case where they do suffice, however.

It should be mentioned that one of these two formulas is due originally to G. HIRSCH [7], who gave it under more restrictive hypotheses than we do. Apparently his proof of this formula is different from ours.

The second part of this paper is concerned with the problem of determining the cohomology ring of a sphere bundle whose characteristic class vanishes. This special case is simpler than the general case, and our results are correspondingly more complete. In this case the structure of the cohomology ring of the bundle may be expressed succinctly by the statement that it is a quadratic extension of the cohomology ring of the base space.<sup>3</sup> From this it follows that the

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<sup>2</sup>The triple product is a secondary cohomology operation on three variables which assigns to any elements  $u \in H^p(B)$ ,  $v \in H^q(B)$ , and  $w \in H^r(B)$  such that  $u \cdot v = v \cdot w = 0$  a coset of a certain subgroup of  $H^{p+q+r-1}(B)$ . This coset is denoted by  $\langle u, v, w \rangle$ . For the definition and properties, see [16].

<sup>3</sup>This fact has apparently been known for some time, although never stated explicitly in this form; see, for example, LIAO [10], or HIRSCH [5].

The definition of a quadratic extension of a ring is the obvious generalization of the definition of a quadratic extension of a field; see N. BOURBAKI's *Algebre*, chap. II, §7. For our purposes, the definition of BOURBAKI needs to be modified slightly because we are dealing with graded, anti-commutative rings.