

# *Fractional Integrals on $n$ -dimensional Euclidean Space\**

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**1. Introduction.** In their study of fractional integrals, HARDY & LITTLEWOOD, [3], considered the operator  $U_\lambda$ , acting on functions in the interval  $(0, \infty)$ , given by

$$U_\lambda f = \int_0^\infty \frac{f(y)}{|x - y|^\lambda} dy, \quad 0 < \lambda < 1.$$

They proved the following theorem:

**Theorem A.** *If  $f$  is a function in  $L^p(0, \infty)$ ,  $1 < p < \infty$ , and  $p < q < \infty$ , with  $1/q = (1/p) + \lambda - 1$ , then  $U_\lambda f$  exists almost everywhere and*

$$\left( \int_0^\infty |U_\lambda f|^q dx \right)^{1/q} \leq A \left( \int_0^\infty |f(x)|^p dx \right)^{1/p},$$

where  $A$  is independent of  $f$ , but may depend on  $p$  and  $q$ .

This theorem has been extended to  $n$  dimensions by SOBOLEFF, [5], and THORIN, [6]. Before stating their generalization, we recall some standard notation.

We let  $x = (x_1, x_2, \dots, x_n)$  denote a general point in Euclidean  $n$ -space  $E^n$ ,  $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ , and  $dx = dx_1 \dots dx_n$  the standard Euclidean Lebesgue measure.

We let

$$T_\lambda f = \int_{E^n} \frac{f(y)}{|x - y|^\lambda} dy, \quad 0 < \lambda < n.$$

The statement of the Soboleff-Thorin theorem is as follows:<sup>1</sup>

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<sup>1</sup>The case  $n = 1$  of Theorem A\* is stated for the interval  $(-\infty, \infty)$ , while Theorem A is stated for the interval  $(0, \infty)$ . A simple argument, however, shows that these two cases can be reduced to each other.