

Linear Growth, Birth and Death Processes

SAMUEL KARLIN & JAMES MCGREGOR

Communicated by W. FELLER

A birth and death process is a stationary Markoff process $x(t)$ whose state space is the non-negative integers and whose transition probability matrix

$$P_{ij}(t) = \Pr \{x(t) = j | x(0) = i\}$$

satisfies the conditions (as $t \rightarrow 0$)

$$(1) \quad P_{ij}(t) = \begin{cases} \lambda_i t + o(t) & \text{if } j = i + 1 \\ \mu_i t + o(t) & \text{if } j = i - 1 \\ 1 - (\lambda_i + \mu_i)t + o(t) & \text{if } j = i \end{cases}$$

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$ and $\mu_0 \geq 0$. The process is called a linear growth process if $\lambda_n = \lambda n + a$ and $\mu_n = \mu n + b$ with $\lambda > 0$ and $\mu > 0$. Such processes occur naturally in the study of biological reproduction and population growth. If the state of the system n describes the size of the population, then the average instantaneous rate of growth is $\lambda n + a$. Similarly, the probability of the state of the process decreasing by one after the lapse of a small period of time is $(\mu n + b)t + o(t)$. The factor λn represents the growth of the population due to the current size of the population. The second factor a may be interpreted as the increase of the state of the system due to an external source. The two components μn and b which compose the death rate possess an analogous meaning.

These birth and death processes represent the most elementary formulations of probabilistic models describing biological growth. Various aspects of these processes have received considerable attention in the recent literature [4] (see page 407). Nevertheless, a thorough understanding of the nature of these simple models is still lacking. For a complete set of references and a lucid survey of the subject matter of linear growth processes, the reader should consult [11] and the extensive bibliography quoted therein.

Our approach to the study of linear growth models derives basically from our knowledge of the elaborate structure of birth and death processes as developed

Prepared under contract Nonr-225(28) (NR-047-019) for Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.