

Some Extremal Problems for a New Class of Univalent Functions

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1. Much attention in the past has been devoted to the study of various extremum problems for the class V of functions $f(z)$ regular and univalent in the unit circle $|z| < 1$ and normalized so that $f'(0) = 1$ and for its different subclasses, for example, the class V_R of real univalent functions which lead to symmetric domains, the class of functions for which the image domains are convex or star-like and the class of bounded univalent functions whose image domains lie within a fixed circle. In the present paper we consider problems for another class S of regular univalent functions $f(z)$ which are real on the real axis and whose image domains contain the unit circle, the functions being normalized so that $f(0) = 0$ and $f(1) = 1$. Some of the simplest functions belonging to the class S are the identity function z and the Koebe function $K(z) = 4z/(1+z)^2$. We observe that if $f(z) \in S$ then $f(z)/z$ is regular analytic in $|z| < 1$ and hence, by the minimum principle, we have

$$(1) \quad |f(z)| \geq |z|.$$

Thus at a given point $\zeta \in U_z(|z| < 1)$ the identity function $f(z)$ gives the minimum absolute value in the class S .

As a consequence of (1) we have, in the limit when $z \rightarrow 0$, that $|f'(0)| \geq 1$. Further, let $f(z) \in S$ have the following power series expansion in $|z| < 1$:

$$(2) \quad f(z) = a_1z + a_2z^2 + a_3z^3 + \dots,$$

where a_1, a_2, a_3, \dots are all real. We then see that the function $f(z)/a_1$ belongs to the class V_R of functions $F(z)$ which are normalized so that $F(0) = 0$ and $F'(0) = 1$. Thus an application of the Koebe quarter theorem immediately gives

$$(3) \quad |a_1| = |f'(0)| \leq 4.$$

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