

A Variational Problem Relating to Cooling Fins

R. J. DUFFIN

Communicated by J. B. SERRIN

1. Introduction. The first part of this paper gives an exact solution of a definite mathematical problem concerning cooling fins. Cooling fins are used to conduct heat away from machines to the ambient medium. For example, circular fins are used on the cylinders of air cooled engines. It is not difficult to see that in order to economize on weight the fin should taper, narrowing in the direction of heat flow. The problem arising may be phrased in this way—find the taper which gives the maximum dissipation of heat for a given weight of fin.

The mathematical formulation of this problem involves a differential equation of Sturm-Liouville type, $Lu = 0$. Here u is the temperature, and the differential operator L depends on the taper. The dissipation is given by the integral of u over the surface of the fin.

A criterion for this optimum problem was proposed by E. SCHMIDT [4]. According to SCHMIDT *the temperature should be a linear function of the distance along the fin*. This relation leads to explicit expressions for the taper of a fin. The same criterion has also been applied to cooling spines [5]. A review of the work of SCHMIDT and others is given in the treatise of JAKOB [6]. It seems that the literature does not contain a proof of the Schmidt criterion. SCHMIDT did advance an intuitive argument, but it is not convincing.

In order to find a rigorous treatment, the problem is here recast in a form suitable for treatment by the calculus of variations. Euler equations are obtained by the formal variational methods. The solution of the Euler equations is extremely simple, namely, that the temperature is a linear function. It is then proved directly that this is a maximizing solution. The Schmidt criterion is thereby justified. Also, explicit expressions are given for the taper and for the maximum dissipation.

The second part of the paper concerns the general problem of finding upper and lower bounds for the solution of the Sturm-Liouville equation $Lu = 0$. Several feasible methods of obtaining bounds are indicated. In particular, these methods could be used to calculate the effect of departure from the optimum taper in cooling fins.