

Convergence of Solutions of Parabolic Equations to a Steady State

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Communicated by E. Hopf

Introduction. In this paper we discuss the behavior of solutions of second order parabolic equations when the time-variable t goes to infinity. Let L be a linear parabolic operator of the second order whose coefficients are functions of (x, t) (x is n -dimensional). In Part I we consider solutions of the equation $Lu = f(x, t)$ and in Part II we consider solutions of the nonlinear equation $Lu = f(x, t) + g(x, t, u)$. The domain where the behavior of the solutions is studied is a cylinder with its base on $t = 0$ and which is contained in $t > 0$. In Part III we extend some results of Parts I and II to domains whose cross section at $t = \tau$ becomes unbounded as $\tau \rightarrow \infty$.

In §1 we state the following theorems: (a) If $Lu = f(x, t)$, if $f \rightarrow 0$ as $t \rightarrow \infty$ and if $u \rightarrow 0$ on the lateral boundary of the cylinder, then $u \rightarrow 0$ as $t \rightarrow \infty$ inside the cylinder (Theorem 1). (b) If $Lu = f(x, t)$, if f and the coefficients of L tend to limits as $t \rightarrow \infty$ and if u tends to a limit as $t \rightarrow \infty$ on the lateral boundary of the cylinder, then u tends to a limit v inside the cylinder as $t \rightarrow \infty$ and v is a solution of the elliptic equation $(L' + \partial/\partial t)v = f'$ where f' and L' are the limits of f and L (Theorem 2). The proofs of these theorems are given in §2 and §3. In the proofs we make use of the maximum principle [8], [10], of existence theorems for the Dirichlet problem [4], [11], [13] and of the Schauder type interior estimates for parabolic equations [3].

Theorem 2, in the special case where L is the heat operator and the boundary values on the lateral boundary of the cylinder satisfy certain additional conditions, was proved by TYCHONOFF [14] and by W. FULKS [2]. Both methods, however, are of special nature.

In §4 we extend Theorem 1 to equations of the form $Lu = f(x, t) + g(x, t, u)$ where $|g| \leq \eta |u|$ and η is sufficiently small (Theorem 3). In §5 we generalize Theorem 2 to equations of the form $Lu = f(x, t) + g(x, t, u)$ where $\lim g(x, t, u) = g(x, u)$ satisfies the Lipschitz condition with a small coefficient (Theorem 4). The proofs make use of an existence theorem proved in [5].

Prepared under ONR Contract Nonr-908 NR 041 037 with Indiana University.