

On the Law of the Iterated Logarithm

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The Law of the Iterated Logarithm, as proved by KOLMOGOROFF [1], can be stated as follows:

Theorem A. Let χ_1, χ_2, \dots be a sequence of independent random variables with mean values zero and dispersions equal to b_1, b_2, \dots respectively. Let

$$S_n = \sum_{\nu=1}^n \chi_\nu, \quad B_n^2 = \sum_{\nu=1}^n b_\nu.$$

Under the conditions

$$(1) \quad B_n^2 \rightarrow \infty, \quad |\chi_n| \leq M_n = o\left(\frac{B_n^2}{\log \log B_n}\right)^{\frac{1}{2}}$$

we have

$$(2) \quad \limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} = 1$$

with probability 1.

MARCINKIEWICZ & ZYGMUND [2] have shown the necessity of condition (1) by constructing a sequence of random variables of the form $\{a_{\nu} r_{\nu}(\omega)\}$, where the $r_{\nu}(\omega)$ are the symmetric Bernoulli variables, for which

$$(3) \quad M_n \cong K \left(\frac{B_n^2}{\log \log B_n} \right)^{\frac{1}{2}},$$

$$(4) \quad \limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} < 1.$$

They also have shown that for every sequence of the form $\{a_{\nu} r_{\nu}(\omega)\}$ conclusion (4) with \leq instead of $<$ holds, regardless of the behavior of the M_n .*

The object of this paper is to construct a sequence of independent random variables which satisfy (3) and for which

* In this connection, see the remark at the end of the paper.