

Self-intersections of Immersed Manifolds

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Introduction. An immersion $f: M^k \rightarrow X^{k+r}$ of one C^∞ manifold into a second is called *n-normal* if for each n -tuple of distinct points x_1, \dots, x_n of M with $f(x_1) = \dots = f(x_n)$ the images \tilde{M}_{x_i} of the tangent spaces M_{x_i} under the differential of f have the minimum possible intersection in $X_y, y = f(x_i)$. Explicitly, it is required that $\dim \bigcap_{i=1}^n \tilde{M}_{x_i} = k - (n-1)r$. We will prove that any immersion of a closed manifold can be C_s approximated (any s) by an n -normal immersion.

If $f: M \rightarrow X$ is n -normal and $(f)^n: (M)^n \rightarrow (X)^n$ is the n -fold product map of f , then the restriction F of $(f)^n$ to the subspace of distinct n -tuples of the n -fold product space $(M)^n$ is t -regular in the sense of THOM [7] on the diagonal Δ of $(X)^n$. Then $F^{-1}(\Delta) = \Sigma_n$ is a manifold of dimension $k - (n-1)r$ which we call the n -self-intersection manifold of f . The reason for this terminology is as follows. Let $\pi_1: (M)^n \rightarrow M$ be the projection onto the first factor. Then π_1 restricted to Σ_n is an immersion $\pi_1: \Sigma_n \rightarrow M$ and $\pi_1(\Sigma_n)$ is the set of points of M which are mapped n (or more) to one by f . That is,

$$\pi_1(\Sigma_n) = \{x \in M; \exists x_2, \dots, x_n \in M, \text{ distinct, and } f(x) = f(x_i)\}.$$

If M is closed then Σ_n is closed; if M and X are orientable then so is Σ_n .

Assume now that M , and hence also Σ_n , is closed. Denote the image of the orientation of Σ_n under $\pi_{1*}: H_{k-(n-1)r}(\Sigma_n) \rightarrow H_{k-(n-1)r}(M)$ by Σ_n^* (use coefficients from Z_2 if Σ_n is non-orientable, otherwise use integer coefficients). We are able to compute this class as follows: Consider the composition

$$H_k(M) \xrightarrow{f^*} H_k(X) \xrightarrow{\lambda} H^r(X) \xrightarrow{f^*} H^r(M)$$

where λ is Poincaré duality (using cohomology with compact supports if X is not closed). Let M denote the orientation of M and $\tau = f^*\lambda f_*(M)$. If M is non-orientable then τ is defined with coefficients from Z_2 . Let $\bar{W}^r \in H^r(M)$ denote the r^{th} Stiefel-Whitney class of the normal bundle of M in X . It is an integral class unless this normal bundle is non-orientable or $r = 1$, in which cases it is defined with coefficients from Z_2 . We will prove