

An Elementary Analytic Theory of the Degree of Mapping in n -Dimensional Space

ERHARD HEINZ

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Introduction. Since the publication of BROUWER's fundamental papers¹ much effort has been made in order to establish the principal properties of the degree of a mapping and, in particular, the Brouwer fixed point theorem by analytic methods which do not involve the concepts of combinatorial topology.² The first satisfactory approach to this problem was made by HADAMARD [5] by means of the Kronecker integral. Proofs of the Brouwer fixed point theorem based on the theory of analytic functions were also given later by G. D. BIRKHOFF & KELLOGG [3] and quite recently by SEKI [17]. In [11] NAGUMO deduced the main theorems on the Brouwer degree by using results on implicit functions and a lemma of SARD [14] on the critical points of a mapping. Furthermore, he gives a proof of a theorem, due to LERAY ([8] and [9]), on the degree of the composition of two continuous mappings. From quite a different point of view the theory of the Brouwer degree and also the more general homology theory were developed by DE RHAM [13]. His results are derived from the theory of currents on differentiable manifolds, which is also connected with the work of L. SCHWARTZ on distributions (see [15] and [16]).

In the present paper we shall establish the main properties of the Brouwer degree by a method which is closely related to the work of DE RHAM, but seems to be of a more elementary character. In view of the applications to existence problems of analysis³ we restrict ourselves to the case where the given mapping $y = y(x)$ is defined in the closure of a bounded open set of the n -dimensional space. In §1 we first define the Brouwer degree for continuously differentiable mappings by means of a volume integral (Definition 1) and then extend this concept to the continuous case (Definition 2). The justification of these definitions depends on three simple lemmas (Lemma 1, 2, and 3), the first of which essentially involves an identity which also plays an important rôle in HADA-

¹ See, in particular, [4].

² For the topological methods see ALEXANDROFF-HOFF [1].

³ See LERAY-SCHAUDER [7]. A complete bibliography is found in LERAY [9].