An Extension of a Theorem of Marcinkie-wicz and Some of Its Applications

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1. Introduction. Let \((M, \mu)\) and \((N, \nu)\) be two \(\sigma\)-finite measure spaces, \(\mathcal{S}(M)\) the class of all real-valued simple functions on \(M\) and \(\mathfrak{M}(N)\) the class of all real-valued measurable functions on \(N\).\(^1\) If \(p, q \geq 1\) we say that a linear operator \(T\) mapping \(\mathcal{S}(M)\) into \(\mathfrak{M}(N)\) is of type \((p, q)\) if there exists a positive constant \(A\) such that for each simple function \(f\) in \(\mathcal{S}(M)\)

\[(1.1) \quad ||Tf||_\sigma \leq A \ ||f||_\rho,\]

where

\[||Tf||_\sigma = \left( \int_N |Tf|^\sigma \ d\nu \right)^{1/q},\]

and

\[||f||_\rho = \left( \int_M |f|^\rho \ d\mu \right)^{1/p}.\]

If \(h\) is in \(\mathfrak{M}(N)\), the distribution function of \(|h|\), \(\lambda(y)\), is defined for each \(y > 0\) by

\[\lambda(y) = \nu\{x \in N; |h(x)| > y\}.\]

The operator \(T\) is of weak type \((p, q)\) if there exists a positive number \(A\) such that, for each simple function \(f\) and \(y > 0\),

\[(1.2) \quad \lambda(y) \leq \left[ \frac{A}{y} \ ||f||_\rho \right]^q,\]

where \(\lambda(y)\) is the distribution function of \(|h|\) and \(h = Tf\).\(^2\)

\(^1\) Here \(M\) and \(N\) denote the point-sets and \(\mu, \nu\) the measures on the \(\sigma\)-rings of measurable subsets of \(M\) and \(N\), respectively. Even though, for simplicity, we restrict ourselves to real-valued functions, our results are easily extensible to spaces of complex-valued functions.

\(^2\) The condition of type \((p, q)\) asserts that \(T\), as an operator on \(\mathcal{S}(M) \cap L^p(M)\) into \(L^q(N)\), is bounded. It is easy to check that if \(T\) is of type \((p, q)\) then it is of weak type \((p, q)\); however, the converse is false (see the appendix of this paper).

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