

A Solvable Exceptional Jordan Algebra

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Communicated by N. JACOBSON

In a recent letter from Paris, NATHAN JACOBSON raised the question of the existence of a solvable exceptional Jordan algebra. A fairly obvious candidate for this place in the family of Jordan algebras can be defined as follows. We let \mathfrak{S} be the simple exceptional Jordan algebra of all three-rowed Hermitian matrices with elements in a Cayley division algebra \mathfrak{C} over a field \mathfrak{F} of characteristic not two, and $\mathfrak{R} = \mathfrak{F}[g]$ be the commutative associative algebra of dimension t over \mathfrak{F} consisting of all polynomials $\alpha_1 g + \alpha_2 g^2 + \cdots + \alpha_t g^t$, with coefficients α_i in \mathfrak{F} , such that $g^{t+1} = 0$. We then form the direct product

$$(1) \quad \mathfrak{G} = \mathfrak{S} \times \mathfrak{R}.$$

This is an algebra of dimension $27t$ over \mathfrak{F} , and is a Jordan algebra. Indeed, it is trivial to verify that the direct product \mathfrak{G} , of any Jordan algebra \mathfrak{S} and any commutative associative algebra \mathfrak{R} , is a Jordan algebra. Also $\mathfrak{G}' = \mathfrak{G}^2 = \mathfrak{S} \times \mathfrak{R}^2$, $\mathfrak{G}^{(k)} = [\mathfrak{G}^{(k-1)}]^2 = \mathfrak{S} \times \mathfrak{R}^{2k} = 0$ if $2^k > t$. Hence \mathfrak{G} is a solvable Jordan algebra. We shall show that, *when t is sufficiently large, the algebra \mathfrak{G} is an exceptional Jordan algebra.*

We shall use the method of proof of an earlier note.* The algebra \mathfrak{S} contains three pairwise orthogonal idempotents e_1, e_2, e_3 whose sum, $e = e_1 + e_2 + e_3$, is the unity element of \mathfrak{S} . Then \mathfrak{S} is the vector space direct sum

$$(2) \quad \mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_{12} + \mathfrak{S}_{23} + \mathfrak{S}_{13},$$

where $\mathfrak{S}_i = e_i \mathfrak{S}$ is one-dimensional. The space $\mathfrak{S}_{ij} = \mathfrak{S}_{ij}$ consists of all elements $x_{ij} = x_{ji}$ of \mathfrak{S} such that

$$(3) \quad 2(e_i, x_{ij}) = 2(e_j, x_{ij}) = x_{ij} \quad (i \neq j; i, j = 1, 2, 3),$$

and it follows that

$$(4) \quad 2(e_k, x_{ij}) = 0 \quad (i \neq j; k \neq i, j).$$

Here we are using (a, b) for the product in the Jordan algebra \mathfrak{S} . The subspaces \mathfrak{S}_{ij} of \mathfrak{S} then each have dimension 8 over \mathfrak{F} , and there are also three linear

* *A note on the exceptional Jordan algebra*, Proc. Nat. Acad. of Sciences, **36** (1950), pp. 374-374.