

On the Equation of the Vibrating String

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Introduction. The usual derivation of the equation of the vibrating string starts from Newton's laws and certain assumptions concerning tensional forces. The elegance and the usefulness of the theory suffer under somewhat vague and unnatural regularity conditions.

In this note we shall derive a slightly more general and more intuitive form of the equation from a purely analytical hypothesis which may be described very roughly as follows. We view the "vibrating string" as a one parameter family of functions $u(t, \cdot)$ defined on a fixed interval I and speak of $u(t, \cdot)$ as the "position" of the string at time t . It will be assumed that an acceleration d^2u/dt^2 exists (at least for sufficiently many values of t and at sufficiently many points) and is of the form Ωu where Ω is a linear operator of local character such that $\Omega f(s) \leq 0$ at each point s where f attains a positive local maximum. In other words, at a point where u assumes a positive local maximum the acceleration, if it exists, is directed towards I .

Although this hypothesis is milder than the classical ones it suffices to derive a generalized differential equation describing the motion. In this equation there appear two positive measures, of which one is easily interpretable as physical mass, the other as attractive force. The equation itself may then be viewed as expressing Newton's second law under the action of an inner tension. It seems surprising that *these notions and relations of mechanics appear as inevitable logical consequences of a purely geometric hypothesis.*

Our equation appears as a natural generalization of the classical one. It covers the case in which the masses and forces may vary discontinuously along the string to the extreme of being concentrated at individual points.¹ Such cases are frequently treated in the applied literature, using approximations and passages to the limit. By contrast, we obtain a simple unified theory in which one existence and expansion theorem covers all cases, obviating "transition conditions" *etc.* Furthermore, *boundary conditions* appear as an obvious extension of the differential operator to the endpoints of the interval I (if any).

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¹ As a curiosity we can treat *continuous* motions of a string such that its mass is concentrated at the *rational*s; see [3].