

An Analogue of the Heinz-Hopf Inequality

ROBERT OSSERMAN

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Of the many proofs known for BERNSTEIN's theorem on minimal surfaces, the first one which was based on geometric properties of a finite piece of the surface was due to E. HEINZ [1]. Later, E. HOPF [2] improved this result. It gives a bound on the Gauss curvature at a point of a minimal surface if one can represent the surface in the form $z = f(x, y)$ over a circle $x^2 + y^2 < R^2$ about the given point.

Recently the author [3] generalized BERNSTEIN's theorem to surfaces which need not be of the form $z = f(x, y)$ but which are restricted by the condition that the normals to the surface omit a neighborhood of some direction. In the present paper we obtain a result analogous to that of HEINZ and HOPF for this wider class of surfaces.

Theorem. *Let S be a simply-connected portion of a minimal surface, and assume that there is some fixed direction in space and a number $\alpha > 0$ such that the normals to S all make an angle at least α with the fixed direction. Let p be any point of S and let s be the geodesic distance from p to the boundary of S . Then the Gauss curvature K at p satisfies*

$$(1) \quad |K| \leq h(s, \alpha)$$

where h does not depend on the particular surface, but only on the numbers s and α . In particular, we may use for the function $h(s, \alpha)$

$$(2) \quad h(s, \alpha) = \left(\frac{2}{s} \cot \frac{\alpha}{2} \csc^2 \frac{\alpha}{2} \right)^2.$$

Furthermore, this bound can be improved by using the direction of the normal at p . Namely, if this normal makes an angle ω with the fixed direction, then

$$(3) \quad |K| \leq \left[\frac{R^2 - r^2}{s} \cdot \frac{2(1 + R^2)}{R(1 + r^2)^2} \right]^2$$

where $R = \cot \frac{1}{2}\alpha$ and $r = \cot \frac{1}{2}\omega$.

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