

On Open Mappings in Banach Algebras¹

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We shall be concerned with a Banach algebra \mathfrak{A} with unit e . The needed information concerning the functional calculus, the Gelfand theory, and Fréchet-analyticity will be found in [2]. Suppose that f is a complex-valued function, holomorphic in an open subset U of the complex plane. If a is an element of \mathfrak{A} with spectrum $\sigma(a) \subset U$, we can form the element $f(a) \in \mathfrak{A}$. Denote by $f(\mathfrak{A})$ the set of all $f(a)$ with $\sigma(a) \subset U$. The problem of determining necessary and sufficient conditions that a neighborhood of $f(a)$ be contained in $f(\mathfrak{A})$ is not yet solved, even in the cases $f(\lambda) = e^\lambda$ or $f(\lambda) = \lambda^n, n \geq 2$, an integer. However, rather natural sufficient conditions are known in these cases [3]. The general approach here is used to make the proofs more transparent, and the specialization to these cases is immediate.³

We use the following additional notations. $\mathfrak{E}(\mathfrak{A})$ is the Banach algebra of all bounded operators on \mathfrak{A} . L_a and R_a are those elements of $\mathfrak{E}(\mathfrak{A})$ defined by $L_a(x) = ax, R_a(x) = xa, a, x \in \mathfrak{A}$. $\rho(a)$ is the complement of $\sigma(a)$ in the complex plane, and is the domain of existence of the holomorphic \mathfrak{A} -valued function $R(\lambda, a) = (\lambda e - a)^{-1}$. We will also consider $R(\lambda, L_a)$ and $R(\lambda, R_a)$ in $\mathfrak{E}(\mathfrak{A})$. When λ is a scalar, $(\lambda e - a)$ will be shortened to $(\lambda - a)$.

1. Lemma. $\sigma(L_a) \subset \sigma(a), \sigma(R_a) \subset \sigma(a)$.

Proof. Let $\lambda \in \rho(a)$. Then, in $\mathfrak{E}(\mathfrak{A})$, $\lambda - L_a$ has the two-sided inverse $L_{(\lambda - a)^{-1}}$, so that $\lambda \in \rho(L_a)$. Similarly, $R(\lambda, R_a)$ is $R_{R(\lambda, a)}$ whenever $R(\lambda, a)$ exists.

2. Lemma. Let $a \in \mathfrak{A}$ and $\sigma(a) \subset U$, an open set. Then there is an $\epsilon > 0$ such that if $h \in \mathfrak{A}, \|h\| < \epsilon$, then $\sigma(a + h) \subset U$.

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³ While this manuscript was in preparation, the author discovered another method of attacking this problem which gives less precise results, but has greater scope of application; the present method promises applications to functional calculi, and, in fact, has so been used by J. SCHWARTZ, *Two Perturbation Formulae*, Comm. Pure and Appl. Math., 8, 1955.