

Isoperimetric Inequalities for the Nonhomogeneous Clamped Rod and Plate

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Introduction. It was conjectured by Lord RAYLEIGH [5, pp. 339–340], and first proved by FABER [1] and KRAHN [4], that of all (homogeneous) membranes of a given area the circle has the lowest principal frequency. This result may be taken as a model for a class of inequalities occurring in mathematical physics which are called *isoperimetric inequalities* by PÓLYA & SZEGÖ [8]. Although the above result for the vibrating membrane obviously has no analogue in the case of a homogeneous vibrating string, an analogous result can be stated for the nonhomogeneous string. Indeed, B. SCHWARZ and the author [7, Theorem 2] proved that among all strings of a given length and equimeasurable density, the least principal frequency occurs in the case that the density is symmetrically decreasing about the midpoint of the string (and the maximum principal frequency occurs when the density is symmetrically increasing). In a recent paper [6], SCHWARZ proved a similar theorem for the nonhomogeneous membrane.

In §1 of this paper we consider the transverse vibrations of a thin cylindrical rod of variable density and constant flexural rigidity which is clamped at both ends, and prove a result analogous to that quoted above for the nonhomogeneous string: *Among all rods of a given length and equimeasurable density, the least (greatest) principal frequency occurs in the case that the density is symmetrically decreasing (increasing) about the midpoint.*

In §2 we consider the transverse vibrations of a thin plate of variable density and constant flexural rigidity extended over a plane domain D and clamped along the boundary C . In the case of constant density, and under the hypothesis that the first eigenfunction does not change sign in D , SZEGÖ [9] proved that the lowest principal frequency among all plates of a given area occurs when C is a circle. Under the same hypothesis concerning the first eigenfunction we prove a similar result for nonhomogeneous plates of equimeasurable density. In order to state this result precisely we must define the symmetrical rearrangements of a continuous function $p(x, y)$ defined over $D \cup C$. Let \bar{D} denote the open disk $0 \leq x^2 + y^2 = r^2 < A/\pi$, where A is the area of D , and let \bar{C} be the boundary ($r = R_0$) of \bar{D} . The rearrangement $p^-(x, y) = p^-(r)$ of $p(x, y)$ in symmetrically decreasing order is defined in the closure of \bar{D} as the (unique)