

# On the Foundations of Dynamic Programming

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**1. Introduction.** Let  $X_T$  be the class of all piecewise continuously differentiable functions  $x$  on  $[0, T]$  such that  $x(0) = 0$ , let  $F$  be a function of two real variables satisfying conditions which will be specified in §2, and set

$$(1.1) \quad f(c, T) = \min_{\substack{x \in X_T \\ x(T) = c}} \int_0^T F(x(t), x'(t)) dt.$$

One of the basic problems of dynamic programming is to approximate the value  $p(c, T)$  of the *optimal policy* or *slope function* at the point  $(c, T)$  in terms of  $F$  and the values  $f(c_1, T - \epsilon)$ , for some fixed  $\epsilon > 0$  and a suitable range of values of  $c_1$ . In this formulation  $p$  is the slope function of the field of extremals through  $(0, 0)$ ; however, the considerations which follow can equally well be applied to other fields.

Heuristically, one may use BELLMAN'S principle of optimality as follows. For any  $\epsilon > 0$ ,

$$(1.2) \quad \min_{\substack{x \in X_T \\ x(T) = c}} \int_0^T F(x(t), x'(t)) dt \\ = \min_{\substack{x \in X_T \\ x(T) = c}} \left\{ f(x(T - \epsilon), T - \epsilon) + \int_{T-\epsilon}^T F(x(t), x'(t)) dt \right\},$$

and since  $p(c, T) = x'(T)$  for the minimizing function  $x$ , one approximates the functions of  $X_T$  by setting  $x(T - \epsilon) = c - \epsilon c' + o(\epsilon)$  and expands (1.2) in terms of  $\epsilon$  to obtain

$$(1.3) \quad F(c, T) = \min_{c'} \{ f(c - \epsilon c', T - \epsilon) + \epsilon F(c, c') + o(\epsilon) \}.$$

Ignoring the  $o(\epsilon)$ , it appears plausible that if  $p_\epsilon(c, T)$  is that value of  $c'$  which minimizes the expression

$$(1.4) \quad f(c - \epsilon c', T - \epsilon) + \epsilon F(c, c')$$

then

$$(1.5) \quad \lim_{\epsilon \rightarrow 0} p_\epsilon(c, T) = p(c, T).$$

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